Journal of Statistical Physics, Vol. 48, Nos. 3/4, 1987

Corrigendum to Phase Transitions on Fractal Lattices with Long-Range Interactions[1]

O. Penrose[2]

The following proof of Theorem 2 should replace the one given in the paper. Indeed, it proves a slightly stronger version of the theorem, with the restriction $N \geqslant n$ removed.

As in the proof of Theorem 1, we can write $\mathbf{x}$ in the form

$$\mathbf{x} = \sum_{0}^{q} m^p \mathbf{x}^{(p)}, \qquad \mathbf{x}^{(0)},..., \mathbf{x}^{(q)} \in A \tag{C1}$$

where A is the generating set defined on p. 72 of the paper and q is some nonnegative integer depending on $\mathbf{x}$.

To prove the right-hand inequality in Theorem 2, which is equivalent to

$$R_N(x) \leqslant K_2 N^{1/D} \tag{C2}$$

define k to be the nonnegative integer satisfying

$$n^{k-1} < N \leqslant n^k \tag{C3}$$

and consider the set Y consisting of all points whose position vectors $\mathbf{y}$ have the form

$$\mathbf{y} = \sum_{0}^{k-1} m^p \mathbf{y}^{(p)} + \sum_{k}^{\infty} m^p \mathbf{x}^{(p)}, \qquad \mathbf{y}^{(0)},..., \mathbf{y}^{(k-1)} \in A \tag{C4}$$

where $\mathbf{x}^{(p)}$ is defined to be $\mathbf{0}$ for all $p > q$.

By Theorem 1 each choice of the vectors $\mathbf{y}^{(0)},..., \mathbf{y}^{(k-1)}$ gives a different $\mathbf{y}$, and since there are n ways of choosing each $\mathbf{y}^{(p)}$, the number of points in

[1] This paper appeared in *J. Stat. Phys.* **45**:69–88 (1986).

[2] Mathematics Department, Heriot-Watt University, Edinburgh EH14 4AS, Scotland.

0022-4715/87/0800-0947$05.00/0

Y is n^k. These points are all members of the fractal lattice F, and their Euclidean distances from $\mathbf{x}$ all satisfy

$$\begin{aligned}|\mathbf{y}-\mathbf{x}| &= \left|\sum_0^{k-1} m^p(\mathbf{y}^{(p)}-\mathbf{x}^{(p)})\right| \\ &\leqslant (1+m+\cdots+m^{k-1})\,\rho_{\max} \\ &< m^k\rho_{\max}/(m-1) \qquad \text{(C5)}\end{aligned}$$

where $\rho_{\max}$ is the Euclidean diameter of A; so there are at least n^k points of F within a distance $m^k\rho_{\max}/(m-1)$ of $\mathbf{x}$. It follows, by the definition of $R_N(\mathbf{x})$, that

$$R_{n^k}(\mathbf{x}) \leqslant m^k\rho_{\max}/(m-1) \qquad \text{(C6)}$$

From (C3) and the fact that $R_N(\mathbf{x})$ increases monotonically with N we have

$$R_N(\mathbf{x}) \leqslant R_{n^k}(\mathbf{x}) \qquad \text{(C7)}$$

and from (3.1) and the left-hand inequality in (C3) we have

$$\frac{m^k\rho_{\max}}{m-1} = \frac{n^{(k-1)/D}m\rho_{\max}}{m-1} < \frac{N^{1/D}m\rho_{\max}}{m-1} \qquad \text{(C8)}$$

Combining (C6)–(C8) we verify (C2), with $K_2 = m\rho_{\max}/(m-1)$.

To prove the left-hand inequality in Theorem 2, which is equivalent to

$$R_N(\mathbf{x}) \geqslant K_1 N^{1/D} \qquad \text{(C9)}$$

we note that for every pair of vectors $\mathbf{x}$, $\mathbf{y}$ in F the difference $\mathbf{x}-\mathbf{y}$ can be written in the form

$$\sum_{p\geqslant 0} m^p(\mathbf{x}^{(p)}-\mathbf{y}^{(p)}) \qquad \text{(C10)}$$

and is therefore a member of a new fractal lattice F^* whose generating set A^* consists of all distinct vectors of the form $\mathbf{a}-\mathbf{b}$, with $\mathbf{a}$ and $\mathbf{b}$ in A. Let H be the set consisting of all points in F^* whose Euclidean distance from the origin is less than δ, where δ is a length to be chosen later [Eq. (C16)], and let $n(H)$ be the number of points in H. Assume for the moment that $N > n(H)$, and let l be the nonnegative integer defined by

$$n(H)\,n^{l+1} > N \geqslant n(H)\,n^l \qquad \text{(C11)}$$

Any vector $\mathbf{z}$ belonging to the original fractal lattice F can be written in the form

$$\mathbf{z} = \sum_{0}^{l-1} m^p \mathbf{z}^{(p)} + m^l \mathbf{z}' \tag{C12}$$

where $\mathbf{z}^{(0)},\ldots, \mathbf{z}^{(l-1)}$ belong to A and

$$\mathbf{z}' = \sum_{p \geqslant l} m^{p-l} \mathbf{z}^{(p)} \tag{C13}$$

is a new vector in F. Decomposing $\mathbf{x}$ in the analogous way, we see that the Euclidean distance between $\mathbf{z}$ and $\mathbf{x}$ satisfies

$$\begin{aligned} |\mathbf{z}-\mathbf{x}| &= \left| \sum_{0}^{l-1} m^p (\mathbf{z}^{(p)} - \mathbf{x}^{(p)}) + m^l (\mathbf{z}' - \mathbf{x}') \right| \\ &\geqslant m^l\, |\mathbf{z}' - \mathbf{x}'| - \sum_{0}^{l-1} m^p\, |\mathbf{z}^{(p)} - \mathbf{x}^{(p)}| \end{aligned} \tag{C14}$$

The vectors $\mathbf{z}$ in F fall into two classes: F_1, consisting of those vectors for which $|\mathbf{z}' - \mathbf{x}'| < \delta$, and F_2, for which $|\mathbf{z}' - \mathbf{x}'| \geqslant \delta$. The set F_1 comprises at most $n^l n(H)$ points, since there are n choices for each of $z^{(0)},\ldots, z^{(l-1)}$ and at most $n(H)$ for $\mathbf{z}'$, since $\mathbf{z}' - \mathbf{x}'$ is a member of H. For points $\mathbf{z}$ in the set F_2, we have from (C14)

$$|\mathbf{z}-\mathbf{x}| \geqslant m^l \delta - (1 + m + \cdots + m^{l-1})\, \rho_{\min} \tag{C15}$$

where $\rho_{\min}$ is the least Euclidean distance between points of A. If we now choose δ as

$$\delta = \left(m + \frac{1}{m-1} \right) \rho_{\min} \tag{C16}$$

then (C15) implies

$$|\mathbf{z}-\mathbf{x}| > m^{l+1} \rho_{\min} \tag{C17}$$

for all $\mathbf{z}$ in F_2. Consequently, all the points $\mathbf{z}$ of F for which $|\mathbf{z}-\mathbf{x}| \leqslant m^{l+1}\rho_{\min}$ belong to F_1, and since F_1 comprises at most $n^l n(H)$ points, there are at most $n^l n(H)$ points of F within a distance $m^{l+1}\rho_{\min}$ of $\mathbf{x}$. Thus, it follows from the definition of $R_N(\mathbf{x})$ that

$$R_{n^l n(H)}(x) \geqslant m^{l+1} \rho_{\min} \tag{C18}$$

and hence, using (C11), (3.1), and the monotonicity of $R_N(\mathbf{x})$ as we did at the end of the proof of (C2), that

$$R_N(\mathbf{x}) \geqslant [N/n(H)]^{1/D} \rho_{\min} \tag{C19}$$

The inequality (C19) was derived on the assumption that $N > n(H)$, but since $R_N(\mathbf{x}) \geqslant \rho_{\min}$ for all N, the inequality (C19) also holds when $N \leqslant n(H)$; therefore (C9) holds for all N, with $K_1 = \rho_{\min}/[n(H)]^{1/D}$. This completes the proof of Theorem 2.